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Gauss and the Early Development of Algebraic Numbers

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The William Betz Analysis

The thoughtful reader of Wm. Betz' paper,* "Next Steps in Education and the Teaching of Mathematics" will be impressed with the statesmanlike quality of his exposition. Few of the so-called educational authorities whose policies respecting school mathematics he so frankly challenges will dare break a lance with him. His aggressive championship of a restored primacy for mathematics in our American schools is backed by a wealth of information, abundantly evidenced by his citations and references. In the spirit and in the letter of his statements one cannot fail to sense philosopher, humanist, teacher. His pronouncements take on added power from his language, which is both driving and incisive.

In our judgment, the effectiveness of his thesis is rooted in the assumption that our pre-war curricula have almost uniformly exhibited mathematics so mutilated, enshackled, incomplete, and so full of gaps and dicontinuities, that its true values have largely been shorn away.

"Thus", says he, "there is no continuous growth in which we profess to believe."

S. T. Sanders.

*National Mathematics Magazine, January, 1944.

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON and A. W. RICHESON

Gauss and the Early Development of Algebraic Numbers

By E. T. BELL
California Institute of Technology

(Concluded from February issue)

6. *Complex multiplication.* The last entry in Gauss' Diary (Item 146), dated 1814 Jul. 9, records an unstated connection, discovered by induction, between the theory of biquadratic residues and the lemniscate functions. The putative connection, according to Gauss, is through the total number of solutions, modulo the Gaussian prime $a+bi$, of the congruence $1 \equiv x^2+y^2+x^2y^2$, where $2+2i$ divides $a-1+bi$. Such a connection was explicated and proved from the theory of complex multiplication for the lemniscate functions, by Herglotz in 1921. The proof is quite complicated and long.[44] It is an earlier hint of doubly periodic functions however that is of interest here.

In Article 335 of the *Disquisitiones Arithmeticæ* (1801), introductory to the famous seventh section of the entire work, Gauss states that the principles he is about to explain for the division of the circle may be applied not only to the circular functions, but also to many other transcendental functions,[44] for example those which depend on the (lemniscate) integral

$$\int \frac{dx}{\sqrt{(1-x^4)}},$$

and moreover also to different kinds of congruences. He promises a full account of these functions in a work to be devoted specially to them, and says that congruences will be extensively treated in a sequel to the *Disquisitiones*. Neither project was carried out to the extent of publication.

What Gauss had accomplished but did not publish in the theory of elliptic functions was surpassed and superseded by the independent works of Abel and Jacobi in the 1820's. In his general theory of congruences, at which he was working in 1797-8, Gauss[16] was forestalled by Galois[39] in 1830. Gauss' theory was to have formed the eighth and concluding section of the *Disquisitiones*, but was omitted, partly to save expense in printing. His actual contributions to both theories were recovered from his private papers. They have been minutely analyzed by the editors of his works.

The immediate question here is the possible influence complex multiplication may have had on the early shaping of Gauss' thoughts on algebraic numbers. It has often been remarked that the Gaussian theory of binary quadratic forms might have been specially designed to accommodate the later theory of complex multiplication—a theory which Hilbert considered the most beautiful part of all mathematics. The same theory, specifically the complex multiplication of lemniscate functions, was conjectured by Jacobi[40] in 1839 to have been the true source of the (Gaussian) complex integers $a+bi$, a,b rational integers, which Gauss defined and used in his theory (1831) of biquadratic residues.[31]

As Gauss neither affirmed nor denied Jacobi's conjecture, and as he was in vigorous health when it appeared in print, there is no positive means of deciding whether Gauss maintained his silence because the conjecture was baseless or because it was founded on concealed fact. Gauss, incidentally, seems to have had an aversion to Jacobi as a scientific correspondent, ignoring his young emulator's mathematical letters, occasionally to Jacobi's considerable chagrin. It is somewhat remarkable then that Jacobi, who at first was disinclined to cede Gauss any priority for the basic discoveries in elliptic functions, ended by giving him more than has been claimed or justified by the editors of his works.

Whatever the cogency of Jacobi's suggestion, it seems unlikely that Gauss, even as a young man, would have permitted himself the scandal of the "$1+4i$ and $1-4i$ parts" which Jacobi would foist on him in the passage summarized presently. For at least thirty years, Gauss turned imaginaries over and over in his mind before he would admit that he had finally satisfied himself that they are as legitimate mathematical concepts as negative rational integers. What he might have thought of using imaginary numbers to enumerate the parts of a lemniscate arc, or anything else, must be left, with Jacobi's inexistent "parts", to the imagination. It is clear enough however what Jacobi meant, even if of itself it means nothing. The specious analogy he promoted with so much zeal is at least alluring, and might even make

sense to an intelligence higher than the mathematical. Gauss ignored it.

Jacobi begins by recalling that Gauss had introduced numbers of the form $a+bi$ as moduli or divisors in his researches on biquadratic residues, and that he succeeded thereby in stating for two complex primes of the form $a+bi$ a reciprocity law as simple as that for quadratic residues, the so-called gem of the higher arithmetic. But however simple such an introduction of complex numbers may appear now (1839), Jacobi continues, it belongs nevertheless to the profoundest ideas of science. New concepts do not always present themselves in the simplest guise. He then confesses his disbelief that arithmetic disclosed so deeply hidden an idea: it was drawn from the study of elliptic functions, and indeed from that particular kind furnished by the rectification of an arc of a lemniscate.

In the theory of the multiplication and division of the lemniscate, complex numbers of the form $a+bi$, according to Jacobi, play exactly the same role as ordinary numbers. The analogue in the theory of the lemniscate functions for the expression of the circular functions of n times a circular arc, as rational functions of circular functions of the arc, is the rational expression (in terms of the appropriate functions) for $a+bi$ times the arc of a lemniscate. Division of the circumference of a circle into n equal parts can be effected by solving an equation of degree n; division of a lemniscate "into $a+bi$ parts" leads to an equation of degree a^2+b^2—the norm of $a+bi$. Thus if the circle is to be divided into 15 parts, it is first divided into 5 and into 3 parts, from which $(1/15=(1/3-1/5)/2)$ the required division is found; to divide the lemniscate into 17 parts, it is first divided "into $1+4i$ and into $1-4i$ parts which, combined, give the 17 parts." The thought here, if any, seems to be the seductive misconception that

$$\text{“}(1+4i)(1-4i)=17\text{”}$$

makes $1+4i$ and $1-4i$ cardinal numbers because their product is the cardinal number 17. From this esoteric revelation it follows, according to Jacobi, that a study of the lemniscate integral must of necessity lead to the introduction of numbers $a+bi$ as moduli. But would Gauss even in a nightmare, have seen himself using imaginary numbers for counting?

Although Gauss may have discovered other values of n beyond the $n=5$, recorded in his Diary (Item 60, 19 March 1797), for which division of the lemniscate into n equal parts is possible by a Euclidean construction,[44] analogously to the division of the circle for $n=3,5,17,\cdots$, Jacobi's ingenious conjecture sounds like a typical example of post hoc ergo propter hoc. Gauss, as has been remarked, took no notice of it.

But then, there were a good many perfectly sensible speculations regarding his work which he also ignored.

To give Jacobi's conjecture whatever credence it may merit, it is necessary to recall an episode from the early history of elliptic functions. From this, anyone may form his own opinion as to whether Gauss was led to his complex integers by his work on elliptic functions. The introduction of complex numbers as variables in analysis is an entirely different matter; and it is established that Gauss' study of the lemniscate functions made him suspect that the complex variable is necessary for an understanding of algebraic functions and their integrals. But to confuse the two kinds of use which Gauss made of complex numbers, is to deprive him of one of his greatest achievements. Although some will disagree, the recognition of algebraic integers seems to arithmeticians to have required a more penetrating insight than did the recognition of complex variables. In the second, at least partially, Gauss had several predecessors; in the first, none.

After Abel's *Recherches sur les fonctions elliptiques* (§I-§VII), appeared[41] in Crelle's Journal for 1827, Gauss[42] wrote (30 May 1828) to Schumacher that Abel had forestalled him "by at least a third" and, curiously enough, had used partly the same notations as Gauss himself. In a letter of 18 May 1828 to Abel, Crelle[42] reported that when he had begged Gauss for something on elliptic functions (Crelle had heard that Gauss had been occupied with the theory for more than thirty years), Gauss replied substantially as he had to Schumacher. In addition, he stated that "other occupations" prevented him for the moment from putting his researches into shape for publication. Abel, he said, had followed exactly the same path that he (Gauss) had set out on in 1798. To Gauss it was not astonishing then that the major part of Abel's results were the same as his own. And, he concluded, since Abel's development was carried out with "so much sagacity, penetration, and elegance, I believe therefore that I am relieved of the editing of my own researches."

The critical detail for Jacobi's conjecture is what Gauss meant by "at least a third." Abel's first publication (§I-§VII) contains nothing on complex multiplication. It ends with some special results on the lemniscate functions as simple illustrations of the general theory of elliptic functions developed in the preceding sections. So it is possible but improbable that the remaining two thirds of Gauss' anticipations may have included more on complex multiplication than has been found in his papers—all of it very special.[44] This would be conclusively confirmed if it could be proved that Gauss was aware of the contents of the continuation (§VIII-§X) of Abel's memoir, which

was received for publication by Crelle on 12 February 1828, and which appeared in his journal for 1828.

In §X, Théorème II, Abel states[43] the conditions that an imaginary value of α shall render the equation

$$\frac{dy}{\sqrt{[(1-y^2)(1+\mu y^2)]}} = \alpha \frac{dx}{\sqrt{[(1-x^2)(1+\mu x^2)]}}$$

algebraically integrable: "it is necessary and sufficient that α be of the form $m \pm \sqrt{-1} \cdot \sqrt{n}$, where m and n are rational numbers. In this case the quantity μ is not arbitrary; it must satisfy an equation having an infinity of real and imaginary roots. Each value of μ satisfies the question."

This was the origin of the vast theory of complex multiplication, whose ramifications are not yet all exhaustively explored. Abel illustrates his general theorem by some special cases. No responsible advocate[44] of Gauss' priorities yet has seriously claimed that Gauss preceded Abel in anything approaching this complete generality; nor has anything been recovered from his scientific remains to substantiate such a claim. The closest approximation to a claim is the frequently expressed opinion that, in elliptic functions as in some other departments of mathematics, Gauss probably discovered a great deal more than has yet been found on paper.[44] So for the present at least, Jacobi's conjecture can be ignored as Gauss himself ignored it.

The further history of elliptic functions is irrelevant for Gauss' part in the early development of algebraic numbers. But as he seems to have had a marked partiality for the lemniscate functions and biquadratic residues, some work of Eisenstein's[45] (1846), which disclosed the intimate connection between the two, may be recalled in passing. For sheer ingenuity, if nothing more profound, Eisenstein's deduction of Gauss' law of biquadratic reciprocity from the complex multiplication of the lemniscate functions would be hard to surpass. Except for the case of two real primes, easily handled otherwise, the law is proved for all cases of Gaussian primes. Eisenstein emphasizes that Gauss' partition of a (proper) complex residue system modulo an odd primary complex integer into "quarter-systems" of associates, is basic for his own analysis.

An alternative proof of the biquadratic law, which Eisenstein[46] gave in the preceding year, is even more suggestive, in that the proofs of the laws of quadratic, cubic, and biquadratic reciprocity are unified analytically. The quadratic case, from this point of view, is degenerate, being derived from properties of the circular functions. In the biquadratic case an uneven Gaussian complex prime is the key to the analysis; while for the cubic case, Eisenstein states merely that in-

tegers $a+b\xi$, ξ an imaginary cube root of unity, are required. It was noted earlier[33] that Gauss used these same integers in 1808 to prove (implicitly) the law of cubic reciprocity. He made no claim[44] to have anticipated Eisenstein.

It would be interesting to know why Gauss restricted himself to what are now called Gaussian integers in the work which he himself published, when he was familiar with other algebraic integers as well. Possibly he was less bold than Eisenstein, who admitted that "perhaps one may disapprove of the use of circular and elliptic functions in arithmetical reasoning. But it is to be observed that these functions enter, so to say, only *symbolically*, and that it would be possible to get rid of them entirely without destroying the substance and the basis of the proofs." He then indicates how this may be done for the quadratic case, and states that something similar but more complicated exists for the other two, "and one can say . . . that by no means is the formula for the multiplication of elliptic functions needed. Nevertheless it does not always seem preferable to avoid analytic functions in arithmetical investigations, especially when it is seen a posteriori that they do not enter essentially into the demonstrations, and that they serve only to fix ideas and abridge the conclusions." Similar opinions were expressed by Gauss regarding the use of circular functions in cyclotomy.

Gauss gave as his reason for multiplying proofs of the law of quadratic reciprocity, the hope that methods of proof applicable also to biquadratic residues would emerge. He found such methods; but it is unlikely from what has been recovered from his papers that any of them combined algebraic integers and the complex multiplication of elliptic functions. Eisenstein's inspiration seems to have been his own. Gauss thought highly enough of Eisenstein to write an appreciative preface—the impersonal part of which has become classic—to a collection of his mathematical papers.[47]

7. *Higher Congruences.* Analogies between algebraic equations and congruences noted in the *Disquisitiones Arithmeticæ* led naturally to the "much higher investigation," of which the theory of congruences is only a part, the posthumously published *Analysis Residuorum.* This was to have formed the eighth section of the *Disquisitiones.* From Gauss' Diary, it is known that this "higher investigation" was undertaken in 1797-8. It is in two parts,[48] the first of which is concerned with binomial congruences for a prime modulus. The contents of this part need not be discussed here, except one rather significant detail which seems to have caused Gauss considerable anxiety. The difficulty was of the same genus as one that was to recur in early attempts to

construct an unexceptionable theory of decomposition for algebraic integers.

Gauss showed (Art. 251) that once any one of the e f-nomial periods introduced by him into the theory of binomial equations (as in the seventh section of the *Disquisitiones*) is given, all the periods are obtainable by solving a simultaneous system of e linear congruences for a prime modulus. There is no difficulty if the modulus is not a divisor of the determinant of the system. In the contrary case, the analysis is inapplicable, and must be supplemented by other considerations. Because this exceptional case can occur so rarely, Gauss remarks, he does not linger over it—a curious reason for a mathematician like Gauss to give. In the second part, concerning the general theory of congruences, exceptional cases of a somewhat similar nature arise, and Gauss promises (Art. 363) that "it will be shown below how this difficulty may be obviated." But as the work ends abruptly, uncompleted, in the middle of a congruence, the promise is unfulfilled.

It is generally supposed however that Gauss had in mind as the proposed remedy congruences modulo p^a, $a>1$, p prime, to supplement the case $a=1$ actually investigated. An entry in the Diary (Item 68) for 21 July 1797 makes a suggestion to this effect. But that is all, and there is no record of why Gauss thought his suggestion might do what was required, or how it might be put into effect. In fact he seems to have tired of the subject. The attempt to prepare the general theory of congruences for publication was abandoned. Other interests absorbed his attention, and until a complete draft of the promised "higher theory of congruences" is found—an improbable contingency—it will not be known what goal Gauss had in view. The "higher theory" would have been the natural way for him to follow if he had been seeking ultimately to construct an arithmetic for the integers of cyclotomic fields. There is a mere fragment on higher congruences when the modulus is a composite number.

The extension of congruences as they appear in the *Disquisitiones* is the factorization of polynomials in a single indeterminate, with rational integer coefficients, for a prime modulus. This is the theme of the second part of the *Analysis Residuorum*.[48] Prime polynomials modulo a prime are defined, and their number for a given prime is determined by means of a generating function and an unexplained application of inversion. This result is noted in the Diary (Item 75) under 26 August 1797. More significant from the standpoint of general arithmetic is the proof that the Euclidean algorithm applies to these "modular" polynomials. The units in the theory are readily detected, and the fundamental theorem of arithmetic follows for the polynomials.

As already noted,[39] Galois (1830) forestalled Gauss in this theory. Gauss never published his outline. It was more in the spirit of Dedekind's[49] exposition (1856) than in that of Galois. With the publication of the outline, it was disclosed that Gauss as early as 1797-8 had foreseen the possibility of using the imaginary roots of congruences invented by Galois.

Gauss at the time was thinking intensively of complex numbers in connection with his first proof (1799) of the fundamental theorem of algebra, and he had invented the notation of congruences partly because it suggested fruitful arithmetical analogies with algebraic equations. But, as appears from his correspondence and his published observations on the "metaphysics"—foundations—of mathematics, he meditated on imaginary numbers for many years before he finally felt that he understood them completely. He was satisfied only when (as he asserted) he reached the same interpretation as Hamilton[50] reached six years later (1837), in which complex numbers are replaced by ordered couples of real numbers. So when, in 1797, another might have been tempted to pursue the analogy between congruences and algebraic equations to its extreme, by endowing congruences with imaginary roots, Gauss refrained. He did so, not for lack of capacity, but most likely because he did not then see how such imaginary roots were to be logically justified.

"It is obvious," he says (Art. 338), "that the congruence $\xi \equiv 0$ has no real roots if ξ has no factors of the first degree; but there is no hindrance, if ξ is resolvable into factors of the second, third, or higher degrees, in ascribing to the congruence *imaginary* roots." And he goes on to say that if he were to permit himself a license similar to that indulged in by recent mathematicians, he would introduce such imaginaries, thereby "incomparably" abridging all of his subsequent investigations. But, he concludes, he prefers to deduce everything "from principles."

Galois was unaware of the predestined iniquity of his imaginaries; for he had been dead all forty-four years when Gauss' condemnation first appeared in print. Dedekind's theory,[49] making no use of imaginaries, would have been acceptable to Gauss. The logical difficulties which Gauss might have observed in Galois' imaginaries were removed in 1866 by Serret.[51] There is therefore no longer any licentiousness in indulging in these imaginaries, if desired.

Much later (1920), the serious lacunæ in Gauss' first and fourth proofs of the fundamental theorem of algebra were filled.[52]. It is one of the finer ironies of mathematical immortality, that while Galois' imaginaries as now rigorized[53] are unobjectionable in the increasingly popular intuitionist philosophy of mathematics, even the amended

proofs for the fundamental theorem are not. However, not all mathematicians are intuitionists.

At last, in 1831, Gauss[7] permitted himself to present his complex integers $a+bi$ to the mathematical public. His meticulous scruples had been overcome; he felt that he now thoroughly understood the imaginaries of algebra; the logic in the crucial proofs had been rigidly secured; and he was confident that history would justify him in his unprecedented boldness. He made no mention of his early adventures in Fermat's last theorem, and dropped no hint that he had proved the law of cubic reciprocity[44] twenty three years earlier by the use of quadratic integers other than those he now acknowledged. All the credit for the epoch-making inspiration was bestowed on biquadratic residues.

It was indisputable that the introduction of Gaussian integers induced an astonishing simplification in the theory of biquadratic residues. Disparate results with no evident connection now appeared as necessary consequences of the arithmetic of the underlying complex domain. The units and primes for the new integers were defined; the Euclidean algorithm was shown to hold: the new integers were as legitimate elements of arithmetic as the rational integers themselves. A "boundless new domain" of the theory of numbers had been consolidated with a firmness which must have satisfied even its exacting discoverer.

Students of human nature may be interested to speculate why Gauss omitted any mention of his early work in algebraic integers. Perhaps he had forgotten all about it. Or he may have thought it unworthy of him, in comparison with his published work on biquadratic residues, and wished to be remembered for that alone. If so, he is but one more victim of an inquisitive posterity which, never content with the picture a man wished to leave of himself, publishes all the private papers he neglected to destroy. Though the scholarly emoluments from such indelicacy may be colossal, the scientific gain is in general negligible, and the increase in understanding the outraged man nil. Considered objectively, biography is the meanest form of gossip.

8. *From the inexistent to the existent.* Having formally pronounced the field of algebraic numbers open, Gauss left it. Others might cultivate it, should they find it profitable. Several did, but Gauss never returned to inspect or bless their labors. Perhaps it was as well for his peace of mind that he held himself aloof from the contemporaries of his declining years. If the mere suggestion of imaginary roots of congruences was repugnant to him in 1797, when he was just twenty

years old, it is unlikely that he would have participated with gusto in a bacchanal of inexistent ideal numbers half a century later.

The account of Gauss' active part in the theory of algebraic numbers should close, logically and historically, with the memoir of 1831 on biquadratic residues. To link the work of Gauss to the present, however, it is necessary to connect him with Dedekind and Kronecker, especially Dedekind. The connection is through Kummer. Kronecker was Kummer's pupil at the time when Kummer was creating his theory. Dedekind was inspired by Kummer's theory to invent his own. From Dedekind to abstract algebra the way, though long, is straight and clear. It will be sufficient to reach Dedekind, the last personal pupil of Gauss. The pupil owed nothing to his master in the matter concerned: Dedekind struck out on a way of his own, of which there is not a hint in anything attributed to Gauss.

The obvious problem after Gauss' introduction of complex integers $a+bi$ into the study of biquadratic residues, was to construct an arithmetic for cyclotomic fields, in the hope that it would lead to the discovery and proof of higher reciprocity laws. The problem lay disregarded till the 1840's, when Kummer[54] took it up, and through the creation of his ideal numbers (1845), was enabled (1847, 1858) to state—with certain critical exceptions—a general reciprocity law.[58]

The 1840's also witnessed a resurgence of Fermat's last theorem. Although the theorem was not disposed of, the abortive attempt (1847) of Lamé[55] to find a general proof early attracted attention to Kummer's fundamentally new work. A profitable debate ensued between some of the leading mathematicians of the time. Lamé's correct proof (1839) for seventh powers[56] had impressed Cauchy, Liouville, and others: Lamé now had to be taken seriously. Although Gauss did not participate in the debate over Lamé's mishap of 1847, he let it be known that in his opinion Lamé was the foremost French mathematician of the day. His own rebuff by the seventh powers which had yielded to Lamé may have caused him to overlook Cauchy, whom he did not acclaim.

Kummer also was an early contestant for Fermatian immortality. About 1843 he submitted to Dirichlet what he had convinced himself was a general proof of Fermat's last theorem. Dirichlet[4] detected the fatal oversight. It was true, as Kummer had stated, that an integer in the relevant cyclotomic field could be resolved into a product of indecomposable integers of the field; but it was false that the resolution was unique, as Kummer had assumed. This initial failure was the beginning of Kummer's greatest success. He set himself to obliterate the defect by creating new elements sufficient to ensure unique factorization.

While Kummer was thus engaged, Lamé was endeavoring to solidify his general proof based on the arithmetic of the field generated by a primitive nth root of unity. Thinking he had succeeded, he submitted his purported proof to the Paris Academy, only to be shown by Liouville, and a little later by Kummer, that he had erred. Acknowledging the error, he persisted, but without success. Cauchy[57] then (1847) tried to span the unbridgeable chasm by a flimsy deduction that the Euclidean algorithm for the greatest common divisor applies to all cyclotomic domains. He finally saw that he had been too optimistic, gave a counter-example from the field defined by $x^{23}=1$ for one of his deductions, and incontinently dropped the problem. Kummer had the field to himself.[58]

Fermat's last theorem had discharged the function for which it was destined. Contrasted with what it helped to instigate, it had at last become the isolated proposition of minor interest which Gauss, perhaps prematurely, in 1816 had declared it to be. Still, until it is settled, it cannot be shrugged off.

A chemical analogy for his ideal numbers, which Kummer himself offered to make them seem less supernatural than his contemporaries found them, may assist the understanding of a later generation. The element fluorine had not been isolated in Kummer's heyday—it was not till 1888; yet chemists inferred not only its existence but also its properties from reactions which, unless such an element existed, were unnatural and inexplicable. There is this cardinal distinction, however, between Kummer's creation and an inferred element: nobody will ever isolate an ideal number, for such a number has no existence. This ghostly unreality of Kummer's ideal numbers was the catalyst which, in Dedekind's mind, transformed inexistence to existence.

Kummer did not exhibit his ideal numbers; it is in fact impossible to do so. But he did define their behavior, and this was sufficient. More precisely, he defined "divisibility by a specific ideal number." Algebraic numbers are "existent" numbers. If an algebraic number (of the kind he considered) satisfies one or more congruences, the existent number in question is said to be divisible by a determinate ideal number corresponding to the property that the algebraic number satisfies the congruences. An existent number is considered as a special case of an ideal number. The "multiplication" of these ideal numbers is defined, and if the products of each of two numbers (ideal or existent) by the same ideal number are both existent, the numbers are said to be equivalent. Such equivalence satisfies the postulates for an equivalence relation as in abstract algebra, so that ideal numbers may be separated by it into classes: a class is the set of all those numbers which are rendered existent by one and the same multiplier. No ideal multi-

plier is required for existent numbers; the class of all such numbers is called the principal class. To each principal class corresponds an infinity of composable forms of degree n in n indeterminates with existent coefficients. The number of such classes is finite; it plays a fundamental part in the theory and in its applications to Fermat's last theorem and higher reciprocity laws.

Probably enough has been said to suggest the elusiveness of Kummer's ideal numbers and the perspicacity demanded in using them. Even Kummer made mistakes—not mere slips— in handling his own invention, some of which were not detected until well into the twentieth century. The theory also lacked the complete generality desirable in modern mathematics; it referred only to cyclotomic fields, whereas a theory of algebraic numbers should be applicable to any algebraic number field.

Dedekind created a general theory. (Others also produced theories for any algebraic field; but Dedekind's was the one that survived.) His aim was to operate wholly in the realm of "existent" numbers. He had observed that Kummer's theory could be simplified by an application of the theory of higher congruences. Accordingly he started from higher congruences in his first attempt at a general theory. Almost successful, he could not dispose of certain exceptional cases. He therefore abandoned what he considered the way of calculation and proceeded ab initio from general principles. A right approach was found by analyzing the problem into its simplest elements and discarding all inessential details. But to say this is not to explain how he did it, or to account for his triumph. As Poincaré observed, it is the man, not the method, that solves a problem.

It is of interest that the way of higher congruences which Dedekind first explored was not hopelessly blocked as he supposed. Technically, the obstacles that stopped him were the extraneous common index divisors. The difficulties were overcome in 1926-8 by Ore,[59] who took, instead of congruences modulo a prime p, congruences modulo p^α, α a fixed integer $>\delta$ where p^δ exactly divides the discriminant of the corresponding equation.

9. *New directions.* Some measure of a century's progress in algebra and arithmetic, consequent on Gauss' use of complex integers to impart simplicity and coherence to a problem in rational arithmetic, may be inferred from two statements concerning the intimate connections between arithmetic and other parts of mathematics. Only half a century separates them. The first is from the preface which Gauss wrote (1847) for Eisenstein's *Mathematische Abhandlungen*; the

second, from Hilbert's preface (1897) to his report on algebraic numbers, *Die Theorie der algebraischen Zahlkörper.*

"The higher arithmetic," Gauss observed, "offers an inexhaustible store of interesting truths, which indeed are not isolated, but stand in a close internal connection, and between which, as science increases, always new and wholly unexpected connections are ascertainable."

With this observation of Gauss' in mind, Hilbert (1862-1943) elucidated the "unexpected connections": "Like Gauss, Jacobi and Lejeune-Dirichlet frequently and emphatically expressed their surprise at the intimate connections between questions concerning numbers and certain algebraic problems, in particular the problem of cyclotomy. Today the intrinsic reason for these connections is completely uncovered. *The theory of algebraic numbers* and the Galois theory of equations have their common roots in the theory of algebraic fields, and this theory of number fields has become the most important part of the modern theory of numbers. The merit for having introduced the first germ of the theory of number fields again belongs to Gauss."

Some further remarks of Hilbert's are of special interest in the present connection: "The theory of number fields is a monument of admirable beauty and incomparable harmony. The most beautiful part of this monument seems to me to be the theory of abelian fields and of relative abelian fields. Kummer and Kronecker revealed it to us, the former by his works on the higher laws of reciprocity, the latter by his investigations on the complex multiplication of elliptic functions. The profound insights given us by the work of these mathematicians in this theory, also show us that in this domain of science an abundance of the most precious treasures still lie hidden, promising a rich reward to the seeker who knows the value of such treasures and who practices the art of discovering them with devotion."

Monuments of "admirable beauty and incomparable harmony" are but seldom erected to the living. Even while Hilbert himself was still scientifically vigorous, both arithmeticians and algebraists began to abandon, at least temporarily, the search for the kind of undiscovered treasures which he described; and his own restless interests changed. A quarter of a century after his eulogy of "the most important part of the modern theory of numbers" appeared, the analytic theory of numbers suddenly began to expand as never before; and the older monuments, cast into the shade, were relatively neglected. Simultaneously, abstract algebra also began to flourish luxuriantly, and presently had overspread the fields of its origin in its rapid growth toward new and unexplored territory. Nobody foresaw either de-

velopment in 1897 when Hilbert published his report on algebraic numbers. Hilbert's own work in axiomatics at the turn of the century was partly responsible for the swerve away from the classic past; and yet he did not anticipate it.

If an outstanding instigator of radical progress cannot predict the consequences of his own work, it is futile for others to prophesy. Yet it is hard to suppress the feeling that half a century hence, it may happen that some of the treasures prophesied nearly fifty years ago by Hilbert, will be recognizable in the findings of the present generation of abstract algebraists. If much that is now vivid will then have been absorbed and blurred beyond recognition in a newer mathematics, the labor of perfecting it is not necessarily futile. The advance of mathematics has been like the rhythm of an incoming tide: the first wave, with ever slackening speed, reaches its farthest up the sand, hesitates an instant before rushing back to mingle with the following wave, which reaches a little farther than its predecessor, recedes, mingles with its successor, and so on, till the tide turns, and all are swept back to the ocean to await the next tide. In each surge forward there is some remnant of all the tides that went before, though whatever remains may long since have lost its individuality and be no more recognizable for what it was.

REFERENCES, NOTES

[39] Liouville, 11, 1846, 398-407.

[40] Crelle, 19, 1839, 314.

[41] Crelle, 2, 1827, 101-181.

[42] Niels Henrik Abel, Memorial, etc., 1902, Correspondence, Lettre XXXI (65). (Slightly different in W 3).

[43] Crelle, 3, 1828, 181. *Ibid.*, 195, for Jacobi's partial statement, received for publication, 2 April 1828—seven weeks later than Abel's complete statement.

[44] G. Herglotz, Zur letzten Eintragung im Gausschen Tagebuch. Berichte ... der Sächsischen Akademie ... zu Leipzig, 73, 1921, 271-6.

Gauss' P, Q, p, q are Weierstrass' σ, σ_3, σ_1, σ_2. In the special case of the lemniscate functions, Gauss proceeds (W 3, 411-12) from $Pi\varphi = iP\varphi$, $Qi\varphi = Q\varphi$, $pi\varphi = p\varphi$, $qi\varphi = q\varphi$ to the calculation of $P(a+bi)\varphi$, $\cdots$, $q(a+bi)\varphi$ for $(a,b) = (1,1)$, $(2,1)$, $(3,1)$ that of the P, Q functions for $(a,b) = (1,2),(1,3)$, $(1,4)$, and that of the P function for $(a,b) = (1,n)$, n a positive integer. The calculations are straightforward from the addition theorem for the lemniscate functions. If Gauss knew the general addition theorem for elliptic functions, he did not record it in any of his papers yet examined. If he did not know the general addition theorem, it was impossible for him to get anything approaching Abel's result by the methods he used. Even for the lemniscate functions no general conclusion is stated.

In the matter of the division of the lemniscate, Abel's letter of December, 1826, to Holmboe announces the complete solution: "I have written a big memoir on elliptic functions, which contains certain rather curious things, and which cannot fail, I flatter myself, to fix the attention of the literary world. Among other things it treats the

division of the lemniscate. Ah! how magnificent it is! I have found that by means of compass and straightedge the arc of the lemniscate is divisible into 2^n+1 equal parts, whenever 2^n+1 is prime. The division depends on an equation of degree $(2^n+1)^2-1$. but I have found its complete solution by means of square roots. That at the same time has enabled me to penetrate the mystery which has enveloped the theory of M. Gauss on the division of the circumference of the circle. I see, clear as day, how that comes about."

Regarding speculations as to what Gauss may have known, Schlesinger (W 10, Abt. 2, Abhand. 2, 9), emphasizes the subjective element in conjectural restorations for which there is no documentary evidence. Whether there is any of this element in the claims (such as Jacobi's) made in Gauss' behalf respecting complex multiplication, is for the individual commentators to settle according to their own judgments. In a similar connection, it has been objected by some that there is a great deal more of Fricke than there is of Gauss in the "restored" proof[32]) of the law of cubic reciprocity; and there are several further instances, for example some concerning modular functions.

[45] Crelle, 30, 1846, 191-4.

[46] Crelle, 39, 1845, 179-183.

[47] Mathematische Abhandlungen, u.s.w., von Dr. G. Eisenstein, u.s.w., mit einer Vorrede von Prof. Dr. Gauss, u.s.w., Berlin, 1847.

[48] W 2, 199-211; 212-240.

[49] Crelle, 54, 1857, 1-26. Dedekind reduced the theory to its arithmetical residuum, avoiding Galois imaginaries, also the appeal to the fundamental theorem of algebra, as by Schönemann in 1844 (Crelle, 31, 1846, 269-325.)

[50] Trans. Roy. Irish Acad., 17, 293. There is no hint in Gauss' works (or in his published correspondence) of the current interpretation of negative integers and rational numbers through number couples of positive integers. This seems to have originated in Kronecker's theory of modular systems.

[51] Reproduced in his Cours d'algèbre supérieure, ed. 4, 2, 1879, 122-189.

[52] By Ostrowski; reprinted as a supplement, 18 pp., to W 10, Abt. 2.

[53] And their theory extended, as in Dickson, Linear groups with an exposition of the Galois field theory, Leipzig, 1901, 1-71.

[54] Reference 9, and Crelle, 30, 35, 40, 44; Liouville, 16, 1851, 377-498, especially Arts. VI, VIII.

[55] Liouville, 12, 1847, 137-171, 172-184.

[56] Liouville, 5, 1840, 195-211.

[57] The relevant papers are reprinted in his Oeuvres, 10.

[58] Berichte, Berlin Akademie der Wissenschaften, 1847, 132-9.

[59] Papers in Mathematische Annalen, 96, 1926, 315-52; 97, 1927, 569-598; 99, 1928, 84-117. See also the exposition in Fricke, Lehrbuch der Algebra, 3, 1928.

The Teachers' Department

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Teaching Mathematics to the A. A. F. College Training Detachment (Air Crew)*

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The Army Air Forces College Training Program (Air Crew) has as one of its main objectives an "attempt to diminish individual differences in education background for subsequent aircrew training."† The problem of the college, then, is to determine what deficiencies the student may have and build the courses to correct these deficiencies. To aid in this task the Army makes available to us the scores on the *Aviation Cadet Educational Classification Test* (*Ac 20 A*), which "is a 150-item multiple choice test made up of five parts. The first part consists of 45 mathematics items, the second part of 45 physics items, the third part of 20 history items, the fourth part of 20 geography items, and the fifth part of 20 reading comprehension items."‡ The scores made on the entire test and on each of the five parts are sent with the students. The Army has determined a "Proficiency Score" for each part of the test. In mathematics the proficiency score is 23, in physics 16, in history 13, in geography 14, and in English 10. No indication of the procedure in determining these scores is given, although tables giving the distribution and centiles of the scores for group of 1130 students are included in one of the memorandums. This group took the test at the Psychological Research Unit of the Classification Center at Nashville, Tennessee, in February, 1943. Some of the

*Read before the Mathematics Division of the Society for the Promotion of Engineering Education in Chicago, June 19, 1943.

†F. T. C. Memorandum No. 50-25-1, February 24, 1943, Headquarters Army Air Forces Flying Training Command, Fort Worth, Texas.

‡F. T. C. Memorandum No. 250-25-2, February 27, 1943, Headquarters, Army Air Forces Flying Training Command, Fort Worth, Texas.

proficiency scores are near the median scores, others are above or below.

Another purpose of the training program is to give the student a command of some of the skills and knowledge which he is expected to need in his future training and service in the Army Air Corps. Hence courses are organized with this in mind. There is some variation in the way this demand has been met in our own unit. Some of the departments have considered the task as a definite job to be done and have started with the background the student possesses and have tried to give him that subject matter which will accomplish the desired ends. Other departments, one is tempted to believe, have thought of a college course in that particular field, and have attempted to reproduce such a course under the present conditions. These latter departments have had no end of trouble, not only in teaching the subject matter, but also in motivation and even with classroom discipline. It is no doubt true that one will also find these same variations in going from one detachment to another, even in the same department.

1. *The Background of the Student.* We do not have much detailed data on this. It would probably not be worth too much because the type of person taking pre-flight work seems to be changing rapidly. Most of the college men have gone through already. In an early group of 300 men there were 60 students, mostly college men. However, there were 47 clerks, 22 salesmen, and 20 aircraft workers. The majority of the remainder were machinists or some sort of skilled workmen, although there were a banker, a merchant, a restaurant man, an artist, a mail-carrier, a lawyer, a dairy technician and 14 farmers; 265 of them had had no college mathematics, 4 were mathematics majors, and 29 had had 10 or more semester hours of mathematics. This indicates that we have a wide range of types of students and previous preparation. One would expect that a course like physics, where the teachers and textbook writers assume that the student has had a year or more of college mathematics, would be very difficult to bring down to their level. This is just what has happened in our physics course, and many students refuse to study it because they know that they do not have the background to understand the explanations. Of the groups we have received, the per cent of students proficient in the various subjects varies and seems to be somewhat lower than the group mentioned above who were tested at Nashville. We have made very little use of these results in our teaching so far. Proficiency in English and history will excuse a student from these courses, but such grades do not excuse a student from mathematics, physics, or geography. All students are required to take these courses. A few students seem to know the subject well enough to be excused,

but most of them can derive enough benefit from review and practice to make it worthwhile to have them continue the course.

2. *Curricula of the College Training Program (Air Crew).* The curriculum required by the Army Air Forces for the aircrew students is divided among five subjects or departments: Mathematics, physics, and geography, required of all students, and English (including speech) and history, which are required of those who are deficient in these two subjects. Every student goes to school for five hours and takes five academic subjects, some of which may be electives. In addition, he has approximately three hours of military activities and physical training. One hour of study is required outside of the classroom. In all, this makes a nine-hour day for every student, the maximum permitted by the Army Air Forces regulations.

Nothing will be said here about the subject matter of any of the courses other than mathematics except that the college was directed to make the courses of "college level." There were no other restrictions set up by the Army regarding the level or difficulty of the topics to be discussed. The restrictions in operation are the instructional time available and the limitations of the students we have to work with. More will be said about this later in connection with the mathematics course.

The subject matter to be included in all courses is prescribed by Army directives. Changes have been made already and will probably continue to be made. The mathematics course is 80 hours in length, and includes units in arithmetic, algebra, geometry, and trigonometry, with ten hours of instruction in the use of slide rule. The Army directives list the topics to be discussed in each unit and give the approximate number of hours to be devoted to each. Thus the course is very definitely set up and presumably is almost purely functional. The boys will probably need all the topics discussed. However, they are not always able to learn them thoroughly in the time available.

Quoting from the Army directive: "The topics listed in the following outline are mainly those needed directly by the student as a basis for subsequent training. There are some, however, which are given for background material. The presentation should be from the standpoint of preparation for aircrew training, with as much time as possible spent on supervised drill. Assignment of problems to be done outside of class is recommended. ... Examinations should be of the open-book type. ... The emphasis on theory should be limited to the minimum essentials to enable the students to appreciate the content of the course. Applications of numerical operations should be emphasized whenever possible."*

*F. T. C. Memorandum No. 50-25-1, February 24, 1943, Army Air Forces Flying Training Command, Fort Worth, Texas, page 1.

The unit on arithmetic aims to give the student accuracy and speed in numerical computations and includes the fundamental operations on whole numbers, decimals, fractions, time units, and angle units. It also includes a study of common units of measure in the metric system, percentage, square root, conversion factors, and the use of tables. The purpose of the unit on algebra is "to emphasize the manipulative skills needed for numerical trigonometry, physics, and elementary technical fields", and deals with the fundamental operations, factoring, equations (including the quadratic), ratio and proportion, and graphs. The unit on plane and solid geometry "should be designed to create accurate conceptions of space on the part of the student and to prepare him for trigonometry and certain phases of astronomy; the content should include a treatment of straight lines, planes, dihedral and trihedral angles, and the geometry of the sphere . . . ; proofs should be held to a bare minimum; great emphasis should be placed on the drawing of figures and the making of simple tridimensional paper models."*

The course in trigonometry "is primarily a course in numerical trigonometry. . . . Substantial emphasis on slide-rule computation, stressing applications, especially those involving vector forces and velocities in problems peculiar to the military service; formulas for the solution of the right triangle and the general triangle on the celestial sphere. . . . A major objective of the course is to give the student confidence in the later use of navigation tables, which frequently makes it unnecessary for the navigator to carry out the solution of spherical triangles."†

The college was allowed complete freedom in the choice of textbooks. This was delegated by the dean to the departments concerned. Certain textbooks were suggested by the army directive but there was no compulsion. Since the curriculum was set up in broad outline only, the departments have considerable freedom in choosing the actual content of the courses. The Army is interested in results only and the means by which the ends are attained are not questioned.

The directives imply that the subject matter is to be on the "college level." "The content of the courses shall be selected from any reputable college text."‡ Just what this means, and whether it is possible, might well result in argument. Obviously there are at least two courses that might be followed in selecting the subject matter and textbooks under these conditions. One would be to consider that

*F. T. C. Memorandum No. 50-25-1, *opp. cit.*, page 3.

†*Ibid*, page 3.

‡Flight Table "E", Revised as of February 11, 1943, Army Air Forces, Headquarters Flying Training Command, Fort Worth, Texas, page 6.

"college level" means to follow traditional college standards and objectives. With this point in view the departments would then select a college textbook in mathematics or physics (or the other courses) and proceed to duplicate the teaching and methods followed in regular college classes as nearly as conditions would permit. The second way would be to select subject matter and textbooks which will most likely accomplish the ends which the Army has asked for so specifically. This would mean that the departments would find what the student has to start with and try to build on that and accomplish the ends desired. The first way makes two basic assumptions: (1) that the students are prepared to take a regular college course, and (2) that this college course gives the students the knowledge and the skills they need. These assumptions may well be unfounded in the case of some textbooks and courses.

In the case of mathematics the required subject matter is so varied and covers such a wide range of subjects that it is not possible to select one college text that covers the whole field. It would be necessary to adopt four or five college books to include the entire curriculum. The directive calls for arithmetic, algebra, plane and solid geometry, and plane and spherical trigonometry. For this reason, and also because the mathematics department felt that we should take the students where they were and teach the most useful material possible, we have adopted a single book written for the purpose and covering all of the material called for. This book is *Basic Mathematics for Aviation*, by Frank Ayres. Whether or not the work is as good as regular college courses seems of little importance to us compared with the idea that we are giving these boys things that they will actually need in their future training as air-crew members of the Army Air Forces.

3. *Methods of Teaching.* The methods of teaching which we have followed are affected and in some instances determined by the way the Army does things. For example, grades must be reported each week, and the grade at the end of the month or term is merely the average of these weekly grades. Hence we have taught by the use of a weekly outline and assignment schedule. At the beginning of each week an outline of the text material to be covered, the supplementary material needed, and the tests to be used during the week, etc. are prepared and given to each teacher. This makes it possible to keep all of the classes together and assures us that each class covers the same material. This has had an added advantage when it has been necessary, as it has several times, to shift teachers and classes during a month or period of teaching. Then no class misses any appreciable amount of subject matter.

In addition to the weekly outline and assignment schedule we have used a considerable amount of supplementary material. At any time that the text material has not been adequate for the students' needs we have not hesitated to add other materials. These may consist of extra problems, tables, or explanations. Two illustrations will be cited. The book we use does not contain a table for the natural trigonometric functions. A short table was mimeographed and passed out to the students. Second, the explanation of the problem dealing with landing at alternate or moving airports given in the book is involved and hard to understand; hence, we mimeographed a simpler explanation based on the one given in the War Department Technical Manual, *Air Navigation*, TM 1-205. This supplementary material adds greatly to the effectiveness and efficiency of teaching. There is no need for a teacher to say that certain material was not available.

We have just begun the use of a list of objectives and assignments for each week's work. These are made out by the teachers before the week's work is begun, mimeographed and handed to the students at the first meeting of class. The students then know what is expected of them and the complete assignment before the week starts. These also serve as valuable aids in making tests for any part of the week's work.

The tests are intended to be used as teaching devices as well as a basis for the grades. Since grades must be reported each week, most of the tests are short; usually they are in the form of five or ten-minute tests given at the beginning of the period. These tests are based on the previous lesson with occasional questions going back several lessons. They are mimeographed and given to all the sections which are studying that particular phase of the subject matter. So far we have graded only a few of them uniformly, but there is some understanding among the teachers as to how they are to be graded. A few of the tests are the objective type and are graded automatically by a grading machine. These furnish excellent means of comparing both the classes and the effectiveness of different teachers in handling the same subject matter. However, since we have not analyzed the students' educational background and his intellectual ability very thoroughly, this is not too accurate as a measurement of the teacher's ability. Nevertheless, the results are interesting.

The question of college credit for this course was raised early at our school and we have already provided for granting such credit from the standpoint of our own institution. Hence, on leaving the college, each student receives a certificate showing that he has completed a certain number of class-hours of study in mathematics and the other subjects. In the event that he reëntered this college at a later date,

this could be counted as an elective course. Whether or not he will receive credit in other schools will depend on the school involved. However, it is my understanding that there is to be a joint evaluation committee set up by the American Council of Education and by the War Department which will check courses offered in various colleges and make recommendations as to the credit to be allowed. There are, of course, many arguments on both sides of this question. The courses, especially the one in mathematics, are not advanced enough to carry college credit in the ordinary sense of the word. Much of the subject matter is taught in high school classes of mathematics. It is also true that there are students taking the course who are thoroughly familiar with some of the work before they start. This is true for two reasons; one is that the Army requires all students to take the courses in mathematics, physics, and geography, regardless of their scores on the Army tests, and, two, even those who made low scores on the Army tests may be familiar with parts of the course.

However, the work is not simply arithmetic, algebra, etc., but it also includes applications of these fields to various kinds of flight problems. Those applications are new and give meaning and vitality which the subjects have never had before. For example, under the subject of geometry we discuss the Triangle of Velocities where the problem is to find two out of five of the magnitudes, ground speed, air speed, heading, track, and wind speed from a given direction. These problems are solved in the chapter on geometry by the use of a scale drawing, while later in the chapter on trigonometry they are solved by the laws of sines or cosines. These and other problems in the course are real to the student and they feel that they must know how to solve them in order to learn to fly.

4. *Some Tentative Suggestions.* There are a few suggestions which can be made from our limited experience in working with this problem. These are merely suggestive and are offered with a great deal of timidity. Some will doubtless be revised after more experience with the program.

(1) *Give the students what they need.* These boys have a job to do. It is our duty to find out as nearly as we can just what the job is and the ways in which mathematics can help in its performance. Then we should equip them with the skills and knowledge which they will need and show them, in as far as we can at the present period, how to apply this knowledge to the problems they will have to solve. If we can do this, there is little else for us to worry about.

(2) *Ignore "college standards" and "college courses."* These students do not care much about college credit. For them the equipment

we can give may well be a matter of life or death. If it is useful, the question of whether or not it is above or below college standard fades into insignificance.

(3) *Start where the students are with reference to knowledge of subject matter.* In view of what has been said above, there is no object in trying to teach even essential subject matter if the students do not have the basic knowledge to assimilate it. Hence it is our duty to find what they know and what they lack, and to proceed from there. This will also be a great aid in motivation; if these students are studying things that they feel are essential for their future work and that are within the range of their abilities, there is little trouble in getting them to work on the course.

(4) *Make the objectives and assignments definite and clear.* These students have a great many demands on their time. They are entitled to assignments that are clear and definite. They should know exactly what they are supposed to do for each lesson, and, if possible, why, and what the assignment problems lead to. The objectives for any topic, or day's work, should be well understood by both students and teacher. We have only recently begun the practice of mimeographing the assignments and the objectives for each week and placing them in the hands of the students. This has been well received by the students and it seems reasonable to suppose that these sheets will prove valuable to them. They should also be valuable for the teachers and should make their teaching more accurate as well as more uniform.

(5) *A reliable evaluation program of some sort should be worked out.* We have done very little of a satisfactory nature along this line. The regulations set up by the Army make this a difficult problem to solve satisfactorily. In our unit it is necessary to grade each student from 0 to 100 with 70 as the passing grade. At the end of each month, and at the end of the entire period, the weekly marks are averaged and this goes in as the grade. Thus a grade on a unit or on the whole course cannot be based on a test given at the end of the period. In fact it seems in a sense unfair to give a unit test and use the results for grading during the current week since they have already been graded in earlier weeks on a part of this work and they have no opportunity to improve their grades on that part. However, we have given some unit tests after twenty lessons on a unit; they furnish an excellent means of comparing the groups and the effectiveness of our teaching.

The whole program as here described is still uncertain and in a a state of change. More experience with the program will undoubtedly bring changes and improvements, but we believe that each teacher of

this type of work should be made to feel the responsibility of preparing the students for a definite, highly important task, and that no other idea or conviction should take precedence over this in importance. Under such conditions, teachers, if at all competent, will give the student something that will stand him in good stead for the rest of his period of service, and that we feel will be of definite value to him in helping him render his best service to the post-war world.

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Problem Department

Edited by
E. P. STARKE and N. A. COURT

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscripts be typewritten with double spacing. Send all communications to EMORY P. STARKE, Rutgers University, New Brunswick, N. J.

SOLUTIONS

No. 521. Proposed by *Daniel Arany*, Budapest, Hungary.

Show that $\int_0^1 \cos i\pi x \cos^n \pi x \, dx = 2^{-n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$, where $2k = n+i$.

Solution by *J. Ernest Wilkins, Jr.*, Tuskegee Institute.

The formula as stated is true only when we take its obvious meaning, that $n+i=2k$ is an even integer. When $n+i$ is odd the integral has the value zero. We define

$$F(i,n) = \int_0^1 \cos i\pi x \cos^n \pi x \, dx.$$

An integration by parts shows that

$$F(i,n) = \frac{n}{i} \int_0^1 \sin i\pi x \sin \pi x \cos^{n-1} \pi x \, dx, \quad i \neq 0.$$

The trigonometric identity,

$$\sin i\pi x \sin \pi x = \tfrac{1}{2}[\cos(i-1)\pi x - \cos(i+1)\pi x],$$

shows that we have the recursion formula

$$F(i,n) = n[F(i-1, n-1) - F(i+1, n-1)]/2i, \quad i \neq 0. \tag{1}$$

Similarly,

$$F(0,n)=\int_0^1 \cos^n \pi x\, dx=(n-1)\int_0^1 \sin^2\pi x \cos^{n-2}\pi x\, dx,$$

whence, upon replacing $\sin^2\pi x$ by $1-\cos^2\pi x$, we have

$$F(0,n)=(n-1)F(0,n-2)/n, \quad n>1. \tag{2}$$

Now from (1), we have $F(i,0)=0$ $i\neq 0$, and we verify directly that $F(0,0)=1$, $F(0,1)=0$, which facts agree with the proposed results. If $i(i\neq 0)$ is a value for which $i+n$ is odd, then $(i+1)+(n-1)$ and $(i+1)+(n-1)$ are odd in (1), and we see that $F(i,n)=0$ as desired. If i $(i\neq 0)$ is a value for which $i+n$ is even, then $(i-1)+(n-1)$ and $(i+1)+(n-1)$ are also even and we have

$$F(i,n)=n\left[2^{-n+1}\binom{n-1}{k-1}-2^{-n+1}\binom{n-1}{k}\right]/2i=2^{-n}\binom{n}{k}.$$

The case $i=0$ follows similarly from reduction formula (2). Thus the induction is complete and the statement is proved.

No. 522. Proposed by *Daniel Arany*, Budapest, Hungary.

Show that

$$\int_0^1\int_0^1 \cos i\pi x\cdot\cos j\pi y[\cos\pi x+\cos\pi y]^n\cdot dx\cdot dy=2^{-n}\binom{n}{k}\binom{n}{m}$$

where $2k=n+i+j$, $2m=n+i-j$.

EDITORIAL NOTE. The method employed by *Wilkins* in the solution of No. 521 above is easily adapted to the present problem. Put

$$F(i,j,n)=\int_0^1\left\{\cos j\pi y\cdot dy\int_0^1 \cos i\pi x[\cos\pi x+\cos\pi y]^n dx\right\}$$

$$=\int_0^1\left\{\cos j\pi y\, dy\cdot\frac{n}{i}\int_0^1 \sin i\pi x\sin\pi x[\cos\pi x+\cos\pi y]^{n-1}dx\right\}$$

$$=\int_0^1\left\{\cos j\pi y\, dy\cdot\frac{n}{2i}\int_0^1 [\cos(i-1)\pi x-\cos(i+1)\pi x]\right.$$

$$\left.[\cos\pi x+\cos\pi y]^{n-1}dx\right\}$$

provided $i \neq 0$. Here the two transformations of the form of $F(i,j,n)$ are obtained by integrating by parts and substituting from the trigonometric identity used in the preceding solution. Thus we have

$$(1) \quad F(i,j,n) = \frac{n}{2i} F(i-1,\ j,\ n-1) - \frac{n}{2i} F(i+1,\ j,\ n-1), \quad i \neq 0$$

Since the order of integration is immaterial, we obtain similarly

$$(2) \quad F(i,j,n) = \frac{n}{2j} F(i,\ j-1,\ n-1) - \frac{n}{2j} F(i,\ j+1,\ n-1), \quad j \neq 0.$$

The identity

$$(\cos \pi x + \cos \pi y)^n = \cos \pi x (\cos \pi x + \cos \pi y)^{n-1} + \cos \pi y (\cos \pi x + \cos \pi y)^{n-1}$$

shows that we have

$$(3) \quad F(0,0,n) = F(1,0,n-1) + F(0,1,n-1) = 2F(1,0,n-1), \quad n > 0.$$

From (1) and (2) we have $F(i,j,0) = 0$ unless $i = j = 0$, but then by direct evaluation we find $F(0,0,0) = 1$, which agree with the proposed formula. If now the proposed formula is valid for a particular $n-1$ and for all i, j, and we substitute it into the right member of (1), (2) or (3), it is easy to show that the left member must also be of the proposed form. Thus we have a complete induction. As in the preceding problem, if $n+i=j$ is an odd integer, the value of the proposed double integral is zero.—E. P. S.

No. 447. Proposed by *D. L. MacKay*, Evander Childs High School, New York City.

Inscribe a triangle ABC in a given circle O so that the projections of B and C on the bisector of angle A and the projection of A on BC are the vertices of a triangle of maximum area.

EDITORIAL NOTE. This most interesting addition to the very few available problems on the maximum of a function of two variables ought not to be dropped without some of the attention it deserves.

Let the bisector of angle A intersect circle O again at N, and let angle AON be represented by θ. (Thus θ is the supplement of $B-C$). Let B', C' be the projections of B, C, respectively upon AN, and let A' be the projection of A upon BC. Suppose $b > c$. (If $b = c$, triangle $A'B'C'$ has the area zero.) From the figure, $AC' = b \cos \frac{1}{2}A$ and $AB' = c \cos \frac{1}{2}A$, whence $B'C' = (b-c)\cos \frac{1}{2}A$. If r is the radius of circle O, then $b = 2r \sin B = 2r \sin \frac{1}{2}(\theta + A)$ and $c = 2r \sin C = 2r \sin \frac{1}{2}(\theta - A)$, so that

$$B'C' = 2r \cos \tfrac{1}{2}A [\sin \tfrac{1}{2}(\theta + A) - \sin \tfrac{1}{2}(\theta - A)] = 2r \sin A \cos \tfrac{1}{2}\theta.$$

Also $AA' = c \sin B = 2r \sin \frac{1}{2}(\theta - A)\sin \frac{1}{2}(\theta + A) = r(\cos A - \cos \theta)$, so that the altitude from A' in triangle $A'B'C'$ is

$$AA' \cos \tfrac{1}{2}\theta = r(\cos A - \cos \theta)\cos \tfrac{1}{2}\theta.$$

Hence the area of triangle $A'B'C'$ is

$$(1) \qquad K = r^2 \sin A \cos^2 \tfrac{1}{2}\theta(\cos A - \cos \theta)$$
$$= \tfrac{1}{2}r^2 \sin A(1 + \cos \theta)(\cos A - \cos \theta)$$

The necessary conditions for maximum K are $\partial K/\partial \theta = \partial K/\partial A = 0$. Thus we have

$$\partial K/\partial \theta = \tfrac{1}{2}r^2 \sin A \sin \theta(1 - \cos A + 2 \cos \theta) = 0,$$

$$\partial K/\partial A = \tfrac{1}{2}r^2(1 + \cos \theta)(2 \cos^2 A - \cos A \cos \theta - 1) = 0.$$

Neglecting factors for which $\theta = 0$ or π, or $A = 0$, (whence $K = 0$), we obtain easily $\cos \theta = -1/6$, $\cos A = 2/3$, from which the triangle is easy to construct.

Further calculations give maximum $K = 25r^2\sqrt{5}/216$,

$$\tan B = -\sqrt{5}(\sqrt{7} + 2)/3, \quad \tan C = \sqrt{5}(\sqrt{7} - 2)/3,$$

$$a = 2r\sqrt{5}/3, \quad b = r(\sqrt{70} + \sqrt{10})/6, \quad c = r(\sqrt{70} - \sqrt{10})/6.$$—E. P. S.

W. B. Clarke points out that triangle $A'B'C'$ is always similar to triangle ABC with ratio of similitude equal to $\sin \frac{1}{2}(B - C) = \cos \frac{1}{2}\theta$, and obtains a formula equivalent to (1) for the area K.

No. 530. Proposed by *H. T. R. Aude*, Colgate University.

There exists an infinite number of sets of three integers or triads where each set may represent the sides of a triangle with only one angle equal to 60°. If for a given interval of the integers several such triads exist, then the one which corresponds to that triangle for which the angles are the closest to being equal we may call the "almost equiangular triangle" for that interval.

Under this assumption, remembering that but one angle is equal to 60°, find the almost equiangular triangle (a) for the interval 1 to 100; (b) for the interval 800 to 1600.

Solution by *Earl V. Greer*, Bethany-Peniel College, Bethany, Oklahoma.

(a) Let $A = 60°$ and let B be the greater of the other two angles. From the law of cosines, $a^2 = b^2 + c^2 - bc$, which may be put in the form

$$(1) \qquad 4a^2 - 3(b - c)^2 = (b + c)^2.$$

If the positive integers $x=b-c$ and $n=2a-(b+c)$ are introduced, (1) becomes

$$a=(n^2+3x^2)/4n, \tag{2}$$

and, since $a<100$,

$$x^2<(400n-n^2)/3<(400n)/3. \tag{3}$$

From Mollweide's equation and (2),

$$\frac{2}{\sqrt{3}}\sin\frac{B-C}{2}=\frac{b-c}{a}=\frac{x}{a}=\frac{4nx}{n^2+3x^2}=\frac{4y}{3y^2+1},$$

where $y=x/n$. Hence $B-C$ is least when y is greatest.* Now from (3) $y<20/\sqrt{3n}$; therefore for $n=1$, y is less than 12, while for $n>1$, y is less than 9. Hence y is greatest when $n=1$ and $x=y=11$. Upon substitution these values lead to $a=91$, $b=96$ and $c=85$.

(b) Exactly analogous argument for the interval 800 to 1600 gives the values $a=1519$, $b=1541$, and $c=1496$.

Also solved by the *Proposer*.

No. 531. Proposed by *Nev. R. Mind*.

If a circle has its center on an equilateral hyperbola and passes through the point of the hyperbola diametrically opposite to that point, then the remaining three points of intersection of the two curves form an equilateral triangle. A geometric proof is desired.

I. Solution by *J. S. Guerin*, Catholic University of America.

Lemma. *Any chord of a rectangular hyperbola subtends equal or supplementary angles at the extremities of any diameter.* (C. Smith, *Geometrical Conics*, Art. 120).

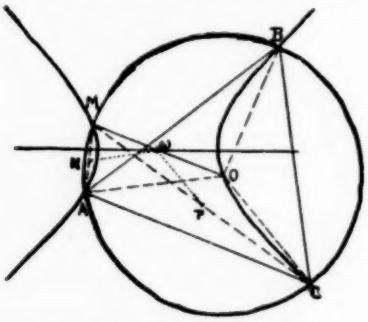
Proof. A and C being the extremities of a chord, and $O\omega M$ a diameter, join OA, OC, MA, MC. Then bisect MA in N and MC in P, and join ωN and ωP. The angle AMC is equal or supplementary to the angle $N\omega P$ between the respective conjugate diameters ωN and ωP. But $\measuredangle N\omega P=\measuredangle AOC$ since their sides are parallel each to each and have the same orientation. Consequently, the angles AMC and AOC are equal or supplementary.

Consider now the hyperbola of diameter $O\omega M$ and intersecting the circle (O) in M, A, B, C. By applying the preceding proposition,

*This statement assumes that $y>1/\sqrt{3}$ which corresponds to the maximum value of $4y/(3y^2+1)$. Actually $y>1$, because $b>a$ implies $b-c>2a-(b+c)$ or $x>n$.

and remarking that $\measuredangle BAC = \measuredangle BMC$ and $\measuredangle AMC = \measuredangle \frac{1}{2}AOC$ and $\measuredangle BAC = \measuredangle \frac{1}{2}BOC$, it is found that $\measuredangle AMC + \measuredangle AOC = (3/2)\measuredangle AOC = \pi$ and $\measuredangle BAC + \measuredangle BOC = (3/2)\measuredangle BOC = \pi$

Hence $\measuredangle AOC = \measuredangle BOC = \measuredangle AOB = (2\pi/3)$.



Therefore, the triangle ABC is equilateral.

Note. The two pencils

$$O(ABC\cdots) \overline{\wedge} M(ABC\cdots)$$

which project the points on the equilateral hyperbola from the two diametrically opposite points O, M on the curve are inversely equal (L. Cremona, *Projective Geometry*, Art. 395, p. 277. Oxford, 1913), hence angle $(OA, OB) =$ angle(MB, MA) both in magnitude and in sign, which provides another proof of the Lemma.—N. A. C.

II. Solution by *L. M. Kelly*, United States Coast Guard Academy.

If the vertices B and C of a triangle ABC inscribed in an equilateral hyperbola (or any hyperbola for that matter) are on one branch and A on the other, the points of the arc between B and C are interior to the triangle ABC and these are the only points of the curve which are. This granted we may proceed as follows:

(1) O is the circumcenter of the triangle ABC. If we can show that it is likewise the orthocenter we have established the proposition.

(2) Note that the center ω of the hyperbola is a point of the nine-point circle of triangle ABC, and also that the orthocenter H must be on the hyperbola. (John Casey, *Analytical Geometry*, p. 290, Dublin, 1893.)

(3) Furthermore the reflection of H in ω must be on the circumcircle, since the nine-point circle bisects all lines joining the orthocenter to all points on the circumcircle.

(4) Properties (2) and (3) lead to the conclusion that H is the reflection of A, B, C or M. We shall prove it is M.

(5) Similar reasoning could be applied to the triangles ABM, ACM, and BCM. Hence A, B, C and M are the reflections of the four orthocenters of these four triangles.

(6) But the reflection of M is O which is also the common circumcenter. Thus one of the orthocenters coincides with the circumcenter and at least one of the triangles is equilateral.

(7) Our original assertion indicates that O is clearly exterior to triangles ABM and ACM, and thus neither can be equilateral.

(8) If BCM were equilateral, ωO would be the perpendicular bisector of BC, which could only happen if ωO were the semi-transverse axis, but then A and M would coincide. The theorem follows.

Also solved by *Henry E. Fettis*, *D. L. MacKay*, and *Col. Harry Jones* who based his solution on a note by *R. C. Yates*, in the *American Mathematical Monthly*, Vol. 50, No. 8, October, 1943, p. 505.

Bibliographical Note by *D. L. MacKay*. This question was first proposed by *Asparagus* in 1884 in *Educational Times*, London, as Q. 7816, and both an analytic and a synthetic proof were published the same year (*Educational Times, Reprints*, Vol. 43, 1885, p. 58).

The proposition was reproduced in *Nouvelles Annales de Mathématiques*, 1884, p. 448, Q. 1507, and an analytic solution published the following year, 1885, p. 382.

It reappeared, in the same periodical, in 1892, as Q. 1641, proposed by G. Lemaire. A synthetic solution was offered in the same volume (p. 46 of supplement at the end of the volume) by H. Brocard who used the occasion to claim priority of the proposition which he made known as early as 1784 in *Mémoires d'Alger*.

Four years later the same *Nouvelles Annales de Mathématiques* published an additional analytical proof of the proposition (1896, p. 146).

EDITORIAL NOTE. It may of interest to recall in this connection another common property of the equlateral hyperbola and the circle: "If two of the four points of intersection of an equilateral hyperbola with a circle are collinear with the center of the hyperbola, the remaining two points are collinear with the center of the circle."

This proposition was considered in *Mathesis*, 1928, p. 136, Q. 2423 and a considerable number of proofs, both analytic and synthetic, were offered. The property is also mentioned by Casey, p. 291, No. 6, of the work cited by Kelly above.—N. A. C.

No. 532. Proposed by *E. P. Starke*, Rutgers University.

For the hyperbolic spiral $\rho\theta = a$, determine (1) the asymtote; (2) the point farthest below the polar axis and the point farthest to the left of the 90°-axis.

Solution by the *Proposer*.

ρ approaches infinity only as θ approaches zero. Then $x=\rho\cos\theta$ approaches infinity and $y=\rho\sin\theta=a\sin\theta/\theta$ approaches a. Hence the spiral has a horizontal asymtote $y=a$.

For the farthest point below the polar axis, y must take a minimum value. Following the customary procedure we have

$$dy/d\theta=a(\theta\cos\theta-\sin\theta)/\theta^2=0 \quad \text{or} \quad \tan\theta=\theta.$$

Values of θ which satisfy this give maximum and minimum points on each turn of the spiral. But as θ increases, ρ diminishes, whence the farthest point below the polar axis corresponds to the minimum value of y for θ between π and 2π. Using tables and the method of interpolation we find easily $\theta=4.494$.

Similarly for the farthest point to the left of the 90°-axis,

$$dx/d\theta=-a(\theta\sin\theta+\cos\theta)/\theta^2=0 \quad \text{or} \quad \cot\theta=-\theta,$$

with θ between $\pi/2$ and $3\pi/2$. Solving, we have $\theta=2.798$.

Also solved by *Jessie E. Swanson*.

No. 535. Proposed by *Pvt. L. Lawrence*, A. S. T. P., Rutgers University.

A marble starts rolling at the top of a sphere under the influence of gravity. Where will it leave the surface of the sphere? (Neglect friction and air resistance.)

Solution by *W. P. Berggren*, University of California College of Agriculture.

Let R represent the radius of the sphere, and r and m the radius and mass of the marble. The energy condition after the center of the marble has traversed an arc θ (of radius $R+r$) will be

$$mg(R+r)(1-\cos\theta)=(7/10)mv^2.$$

In this equation the left member is the potential energy lost and the right member is $\frac{1}{2}mv^2+\frac{1}{2}I\omega^2$, the kinetic energy gained, where $I=(2/5)mr^2$ and $v=r\omega$. The velocity of the marble at any position on the contact arc is then given by

$$v^2=(10/7)g(R+r)(1-\cos\theta).$$

The normal force holding the marble and sphere in contact consists of static and centrifugal components:

$$N=mg\cos\theta-mv^2/(R+r).$$

Elimination of v^2 between these two equations gives

$$N=(17\cos\theta-10)mg/7.$$

The marble will leave the spherical surface where $N=0$, after traversing an arc $\theta=\cos^{-1}(10/17)$, approximately 54°.

Julius S. Miller, in giving a solution essentially equivalent to the above, points out that if there were no friction between the sphere and the marble, the marble would slide only, not roll. The effect on the above analysis would be to remove the term $\frac{1}{2}I\omega^2$ used in computing kinetic energy. The value for θ would then be $\cos^{-1}(2/3)$. To obtain the result $\cos\theta=10/17$, we assume rolling without slipping right up to the moment the marble leaves the sphere, but this would require an infinite coefficient of friction. In any real situation, then, the marble will leave the sphere after traversing an arc somewhere between the two values of θ given. The greater the coefficient of friction the longer the marble remains on the sphere.

Also solved by *Fred Fender, L. M. Kelly, Frank Hawthorne* and *J. N. Snyder, Jerzy Szmojsz*, and *W. I. Thompson.*

No. 537. Proposed by *E. P. Starke*, Rutgers University.

There are just two values of $n(n>10)$ for which the sum of the squares $10^2+11^2+\cdots+n^2$ is exactly divisible by the sum of the numbers $10+11+\cdots+n$. Find them.

Solution by *Earl V. Greer*, Bethany-Peniel College, Bethany, Oklahoma.

Since $1^2+2^2+\cdots+n^2=n(n+1)(2n+1)/6$, we can write

$$\frac{10^2+11^2+\cdots+n^2}{10+11+\cdots+n}=\frac{2n^3+3n^2+n-1710}{3(n-9)(n+10)}$$

$$=\frac{2n^2+3n+190}{3(n+10)}=\frac{1}{3}\left(2n+1+\frac{180}{n+10}\right),$$

which is an integer only if $n+10$ is a factor of 180. For $n=170$, this quotient becomes 113; for $n=35$, it is 25.

Also solved by *J. Ernest Wilkins, Jr.*

PROPOSALS

No. 560. Proposed by *N. A. Court*, University of Oklahoma.

A plane passing through a given point F cuts the lines DA, DB, DC in the points P, Q, R, and the six points P, Q, R, A, B, C lie on a sphere (S). If the plane ABC varies, remaining parallel to itself, what will be the locus of the center of the sphere (S) ?

No. 561. Proposed by *Julius S. Miller*, Dillard University, New Orleans.

A smooth homogeneous sphere rolls, without slipping, on a plane surface. Find the relationship between the kinetic energy of the upper half and that of the lower half.

No. 562. Proposed by *Frank C. Gentry*, University of New Mexico.

A variable triangle has a fixed base, and the difference of the base angles is constant. Find the locus of the nine-point center.

No. 563. Proposed by *E. P. Starke*, Rutgers University.

On the floor ABC of a triangular room is placed a rug with its edges parallel to the sides of the room. Determine the largest such rug which can be turned entirely around on the floor without the walls interfering.

No. 564. Proposed by *Henry E. Fettis*, Dayton, Ohio.

Find the locus of the intersections of the tangents to a conic which make a fixed angle with each other.

No. 565. Proposed by *Fred Fender*, Rutgers University.

The equivalent electrical resistance s of two resistances x and y connected in parallel is given by the relation

$$\frac{1}{s}=\frac{1}{x}+\frac{1}{y}, \quad s, x, y>0.$$

Determine the solutions in positive integers.

No. 566. Proposed by *N. A. Court*, University of Oklahoma.

Construct a skew quadrilateral such that one side shall pass through a given point and shall be perpendicular to the opposite side, and the remaining two sides shall lie on two given skew lines.

Bibliography and Reviews

Edited by
H. A. SIMMONS and P. K. SMITH

Table of Reciprocals of the Integers From 100,009 Through 200,000. Mathematical Tables Project Work Projects Administration of the Federal Works Agency. Arnold N. Lowan, Technical Director. Columbia University Press, New York, 1943 N. Y., viii+201 pages.

This table was computed to supplement the already existing tables of reciprocals (see the Preface for Bibliography) over the interval where these tables were not complete. It will undoubtedly be welcomed by anyone who has to do extensive computation with reciprocals or who prefers to do calculations by multiplication of reciprocals rather than by division. The table gives the reciprocals for each integer within the numbers of the title correct to seven significant figures. Interpolation tables for the next digit are also given. Only the significant figures are given and are attractively arranged with 509 numbers per page in the usual manner of printing tables with the first five digits given in the left hand vertical column and the sixth digit given horizontally in the top row. The tabular differences are given in the extreme right column.

The only explanation of how to use the table is given by two paragraphs which are printed at the bottom of *each* page. This seems to be rather a waste of paper but does serve the purpose of keeping the book the same length and width as the other tables in this series. In an otherwise nicely arranged page, it was surprising to see the interval of numbers for each page obscurely placed in the upper right hand corner in the same size type as the table itself. For convenience of location, it would have seemed advisable to have printed this interval in bold face type at the top of each page; or, perhaps, to have put it at the bottom of the page in place of the repetitious explanations.

Louisiana State University. KAJ L. NIELSEN.

Basic Mathematics for War and Industry. By Paul H. Daus, John M. Gleason, and William M. Whyburn. The Macmillan Company, New York, 1944. xi+277 pages. $2.00.

This book presents the basic facts of arithmetic in 30 pages, of algebra in 53 pages, of plane geometry in 67 pages, of plane trigonometry in 51 pages, and of solid geometry and spherical trigonometry in 45 pages.

The first two chapters instruct the student adequately, we believe, in uses of the four fundamental operations in arithmetic and algebra; and, in the latter subject, the student is carried through proportion, simultaneous linear equations, graphical solution of simultaneous equations, systems of three linear equations, and quadratic equations.

Chapter III gives the more interesting theorems of plane geometry about straight lines, triangles, and circles, and indicates how π arises, through a limiting process.

Chapters IV and V afford the fundamental material of customary courses in plane trigonometry and in spherical trigonometry with applications.

The exposition of the text is clear. One would not ask for better explanation of the principles covered.

The exposition is occasionally novel. For examples of novelty, we mention the following things: the discussion in arithmetic of significant digits, computation with approximate numbers, and reference to computing machines and slide rules; calling an equation in plane trigonometry a *law of cotangents* (p. 193); presenting a diagram in spherical trigonometry in which a line segment is labeled in two ways, as cosa sinb and as tanb cosc (Fig. 249, p. 218),—this being a very helpful device in deriving formulas for the solution of right spherical triangles. It may also be regarded as novel for the book to contain 277 figures.

One of the best things about the book is its problem lists. The problems contain an enormous amount of information from physics, mechanics, engineering, and other subjects. We believe that most any college teacher will find refreshing problems in this text.

The authors state in their preface that "the book should be a suitable text for courses offered under the Engineering, Science, and Management War Training Program; for the various armed service courses both within the army schools and in the colleges; for vocational and other specialized school courses, as well as for home study courses where material is being reviewed or learned for the first time. It should be a useful reference book in connection with science and engineering courses where accurate elementary mathematical principles need to be recalled. . . ."

We agree with the above quotation. We also acknowledge that this book has stirred unrest within us. We feel that we should know how to use a computing machine and that we should know the material that relates to many a diagram of this text, where the diagram appears entirely new to us.

Northwestern University. H. A. SIMMONS.

An Introduction to Navigation and Nautical Astronomy. By W. G. Shute, W. W. Shirk, G. F. Porter, and C. Hemenway. Macmillan Company, New York, 1944. xiv+449 pages. $4.50.

This book is an excellent, teachable, usuable and complete introduction to the subject. Almost every problem which confronts the navigator and pilot is mentioned or discussed or given in full. Throughout the book references are given to larger or special sources pertinent to each subject.

The work is distinctly modern. Very few items of the interesting history of navigation are mentioned and it is relatively non-mathematical in treatment. For example, while the analogous celestial and terrestrial spherical triangles are shown, and their various parts defined, there are no equations given for them. This omission can be condoned, for a navigator today need not know them, nor does he need to use them in doing his work. Everything has been reduced to convenient tables and to graphical methods.

Sample pages and complete tables abound throughout the book. Ageton's simple and fully adequate tables H. O. 211 are given in full. They give the calculated altitude and azimuth of celestial bodies almost by inspection.

Incidentally, it may surprise some of the readers of this journal that the range in time required to solve the navigational, spherical triangle by the half dozen or so modern methods of navigatotin is negligible. An experienced computer has used about 50 seconds for the quickest method and 97 seconds for the longest, in order to get the computed altitude and the azimuth of the sun, moon, planet or star for a given moment, if the observations have been properly prepared before starting to use the tables.

This short interval is, however, the smallest part of the time required.

Thus it may take the experienced navigator and his quartermaster five minutes to make pre-observation computations, fifteen or more minutes to compare hack watch

with chronometer determine the correction to the latter, break out his sextant, check its adjustments and index correction, make the observations and get them averaged and corrected preparatory to using the tables. After that has been done, he must plot his lines of position and finally read the latitude and lognitude of his "fix".

On pages 45 and 46, under the section, entitled Log Book, are given in a group the Beaufort Wind Scale, tables defining States of Weather, Fog and Visibility Scale, State of the Sea, and Character of the Sea Swell. This is an excellent idea.

A fifteen page chapter introduces the student to the *Elements of Aerial Navigation* and solves a few of the more common maneuvering problems graphically.

Like most modern textbooks on navigation, this one does not give the tables from Bowditch called Distance of an Object by Two Bearings, which are most convenient in piloting. This reviewer does not understand why this table is so consistently omitted unless it be because most of the navigator's work in piloting is done graphically as the bearings are laid down on the charts, and because many use the slide rule to solve the plane triangles involved.

This book is confidently recommended for a prominent place in the libraries of those who teach and those who practice navigation.

Dearborn Observatory. OLIVER J. LEE.

Vectors and Matrices. By C. C. MacDuffie. Open Court Publishing Co., LaSalle, Ill., 1943. xi+192 pages. $2.00.

This seventh volume of The Carus Mathematical Monographs series, published by The Mathematical Association of America, is a valuable addition to the expository literature of vectors and matrices and well fulfills the purpose of these Monographs. It will be a part of the working library of every algebraist and should attract a much wider circle of readers.

The central problem of the volume is the rational reduction of a matrix to canonical form under similarity transformations. The first five chapters develop the necessary background for this problem and the seventh chapter applies the result to a discussion of the elementary divisor theory. There then follows a short account of orthogonal transformations, including the principal axis reduction of a quadratic form. The book closes with a self-contained chapter giving the modern description of a vector space as a commutative group with a field as operator domain.

The solution of a system of linear equations is treated in the first chapter without the use of determinants. The author remarks in his introduction that "the importance of the concept of determinant has been, and currently is, vastly over-estimated." The reviewer is in full agreement with this statement and highly recommends MacDuffie's brief treatment to non-mathematical as well as mathematical readers. It should, moreover, be required reading for every author of a textbook in college algebra.

Determinants are adequately treated in Chapter III after the prerequisite material on matrices has been introduced. The treatment is based on the consideration of the determinant as a valuation of a matrix. This approach, while novel in an elementary discussion, has the advantage of being less formal and better motivated than more customary ones.

In most of the book, theorems refer to an arbitrary field, with the necessity for special proofs noted in some places when the characteristic of the field is two. On page 52, however, it seems to be implicitly assumed that the field is algebraically closed. The essential conclusion that the polynomial $dW(l)$ consists of a single term follows without this assumption from the remark that $dW(l^p) = [dW(l)]^p$, at least when the characteristic of the field is not two.

No troublesome errors or misprints were noted. A bibliography and index add to the very considerable usefulness of the book.

WALLACE GIVENS.

Calculus Refresher for Technical Men.. By A. Albert Klaf. McGraw-Hill Book Company, Inc., New York, 1944. viii+431 pages. $3.00.

This book is written in a question and answer style. For examples, Sections 1 and 204 of the book are as follows:

"1. *What is a constant?*
A quantity whose value is fixed.

204. *When is there no maximum or minimum?*
When $(dy/dx)=0$ *yields an impossible result.*"

The answer to the question in Section 204 indicates that the author does not expect the student to be familiar with the plotting of such a curve as that which is represented by the equation

$$y^3=x^2;$$

that is, the class of functions that he would allow is restricted, but in a way that most teachers probably allow in a beginning course. We find in the book numerous places where refinement is desired. However, the main ideas of a beginning course are rather well set forth.

The material covered in the text is nearly all that one finds in a current beginner's book, designed for 10 semester hours of calculus. In applications, this text is most excellent. Any teacher is apt to be refreshed by working through Section III of this book.

Among the main omissions of the book are two fundamental differentiations, namely, those of $\sin x$ and $\log_e x$, and a chapter on series. To differentiate, e^x, the author expands

$$(1+1/n)^{nx} \tag{1}$$

by the binomial theorem, takes the limit of the resulting series as n increases indefinitely, and then differentiates this latter term-by-term. He thus obtains the formula for the derivative of e^x with respect to x. Then, by use of this result, he obtains the derivative of $\log_e x$. The book contains very little about series.

The author's procedure makes for speed; it enables the student to review, or learn for the first time, a large body of facts about calculus in a short time; and, in spite of any lack of rigor—which would be expected in any refresher text on calculus—we believe the book is very worth while as a "refresher". In fact, we would not be afraid to use it as a text for beginners; for, in the very act of pointing out the weaknesses of this text, we would stand a good chance of impressing fine distinctions upon the student, which he would probably not get from one of the more rigorous current calculus texts.

Northwestern University. H. A. SIMMONS.